6. Consider the function:

$$f(x) = \frac{\sin(\frac{x}{3})}{2x}.$$

(a) (5 points) Find a Taylor series centered at $a = 0$ (MacLaurin series) for the function f given above. Simplify your answer and write in $\sum$-notation.

(b) (4 points) Use the series from part (a) to find $f^{(8)}(0)$. You do not need to simplify your final answer.

(a) $2x \cdot f(x) = \sin\left(\frac{x}{3}\right)$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N \left(\frac{x}{3}\right)^{2N+1}}{(2N+1)!} = \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N+1}}{3^{2N+1}(2N+1)!}$$

$$f(x) = \frac{1}{2} \cdot \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N}}{3^{2N+1}(2N+1)!} \quad (*)$$

(b) recall the Taylor series expansion!

$$f(x) = \sum_{N=0}^{\infty} \frac{f^{(N)}(0)}{N!} x^N \quad (**)$$

→ find the coefficient of x^8 in $(*)$ and $(**)$:

in $(*)$: $2N = 8 \longrightarrow N = 4$

$$\frac{(-1)^4}{3^9 \cdot 9!} \overset{\text{by } (**)}{=} \frac{f^{(8)}(0)}{8!}$$

$$\longrightarrow f^{(8)}(0) = \frac{1 \cdot 8!}{3^9 \cdot 9!} = \frac{1}{3^9 \cdot 9} = \frac{1}{3^{11}}$$

4. (a) (12 points) Find the second degree Taylor Polynomial for the function $f(x) = x^{1/4}$ centered at $a = 1$.

(b) (8 points) Estimate $f(0.2)$ for your answer in part (a). Simplify your answer as far as you can without a calculator.

(c) (10 points) Find the maximum error in your approximation in part (b). Simplify as far as you can without a calculator. Recall: $|R_n(x)| \le \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$.

(a) find $P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(1)}{k!}(x-1)^k$ (*)

→ first find the derivatives in the formula:

$f^{(0)}(x) = f(x) = x^{1/4} \longrightarrow f^{(0)}(1) = 1$

$f^{(1)}(x) = \frac{1}{4x^{3/4}} \longrightarrow f^{(1)}(1) = \frac{1}{4}$

$f^{(2)}(x) = -\frac{3}{16x^{7/4}} \longrightarrow f^{(2)}(1) = -\frac{3}{16}$

$\longrightarrow P_2(x) = 1 + \frac{1}{4 \cdot 1!}(x-1) - \frac{3}{16 \cdot 2!}(x-1)^2$

$= 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$ (**)

(b) estimate $f(0.2) = f(\frac{1}{5})$

→ plug in $x = \frac{1}{5}$ into (**) to find $P_2(\frac{1}{5})$

(c) How good/bad was our approx. in part (b)?

→ use the Taylor remainder formula:

$$|R_N(x)| \le \max_{|x-a| \le c} |f^{(N+1)}(c)| \cdot \frac{|x-a|^{N+1}}{(N+1)!}$$

→ here: take $N = 2$, $a = 1$, $x = \frac{1}{5}$

→ Need $f^{(3)}(x)$!

recall from above that $f^{(2)}(x) = -\frac{3}{16x^{7/4}}$

$$f^{(3)}(x) = \frac{21}{64x^{11/4}}$$

$\longrightarrow$ compute:

$$\max_{\frac{4}{5} \le c} \frac{21}{64c^{11/4}} = \frac{21}{64\left(\frac{4}{5}\right)^{11/4}}$$

$$\longrightarrow \left|R_2\left(\frac{1}{5}\right)\right| \le \frac{21}{64\left(\frac{4}{5}\right)^{11/4}} \cdot \frac{(4/5)^3}{3!}$$

5. (16 points) Determine if the series below converges or diverges. Justify your answer using the convergence tests from class. Be sure that you (1) name the test and state the conditions needed for the test you are using, (2) show work for the test that requires some math, and (3) state a conclusion that explains why the test shows convergence or divergence.

$$S = \sum_{k=2}^{\infty} \frac{\ln k}{k^5}$$

- (3 points) State the test and conditions. We will apply the integral test. We need the function to be positive, decreasing, and continuous. If $f(x) = \frac{\ln x}{x^5}$, then $f'(x) = \frac{x^5 \cdot \frac{1}{x} - 5x^4 \cdot \ln x}{x^{10}}$ is negative when the numerator is negative. The numerator reduces to $x^4(1 - 5\ln x)$. We note that $1 - 5\ln x < 0$ when $\ln x > \frac{1}{5}$ or $x > e^{1/5} \approx 1.22$, so the function decreases when $x \geq 2$.

$$S = \sum_{k=2}^{\infty} \frac{\ln(k)}{k^5}$$

→ apply the integral test to check for convergence of S:

→ Conditions to apply the integral test (to $f(x) = \frac{\ln(x)}{x^5}$):

- f(k) has to be positive for $k \geq 2$
- f(x) has to be cts on $[2, \infty)$
- f(x) needs to be decreasing:

Compute $f'(x) = \frac{d}{dx}\left[\frac{\ln(x)}{x^5}\right]$ $\left(\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \right)$

$$= \frac{x^4 - \ln(x) \cdot 5 \cdot x^4}{x^{10}} \underbrace{< 0}_{\text{need this to happen}}$$

$$\Leftrightarrow 1 - 5\ln(x) < 0$$

$$\Leftrightarrow \ln(x) > \frac{1}{5} \Leftrightarrow x > e^{1/5}$$

(This happens when $x \geq 2$)

→ So we have checked the conditions, and we can apply the integral test

→ integral test tells us that

S converges exactly if

$\int_2^{\infty} f(x)\,dx$ converges (and diverges otherwise)

$$I = \int_2^{\infty} \frac{\ln(x)}{x^5}\,dx \qquad \left(\frac{\underline{u\text{-sub:}}}{\substack{u = \ln(x) \\ du = \frac{dx}{x}}} \;,\; \frac{1}{x^4} = e^{-4u} \right)$$

$$= \int_{\ln(2)}^{\infty} u e^{-4u}\,du \qquad \left(\underline{\text{IBP:}} \quad \int u\,dv = uv - \int v\,du \right)$$

$$\begin{bmatrix} u = u & dv = e^{-4u}\,du \\ du = du & v = -\frac{1}{4}e^{-4u} \end{bmatrix}$$

$$= -\frac{1}{4}ue^{-4u}\Big|_{\ln(2)}^{\infty} + \frac{1}{4}\int_{\ln(2)}^{\infty} e^{-4u}\,du$$

$$= -\frac{1}{4}ue^{-4u}\Big|_{\ln(2)}^{\infty} - \frac{1}{16}e^{-4u}\Big|_{\ln(2)}^{\infty}$$

$$= \lim_{x\to\infty} -\cancelto{0}{\frac{1}{4}xe^{-4x}} + \text{constant}$$

$$+ \lim_{x\to\infty} -\cancelto{0}{\frac{1}{16}e^{-4x}} + \text{constant}$$

$$= \text{constant}$$

$\longrightarrow$ So the integral I converges, and this shows that S converges.

(b) (16 points) Determine the radius and interval of convergence of the power series. You may use your answer from part (a) to assist with checking the endpoints.

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{(k^4+5)^{1/5}}(2x-4)^k.$$

$$f(x)=\sum_{k=1}^{\infty}\frac{(2x-4)^k}{(k^4+5)^{1/5}},$$ find radius of conv (R), IC

$\rightarrow$ write the series for $f(x)$ in "standard form", or in powers of $(x-c)$:

$$f(x)=\sum_{k=1}^{\infty}\frac{2^k}{(k^4+5)^{1/5}}(x-2)^k \qquad \left(c=2,\ a_k=\frac{2^k}{(k^4+5)^{1/5}}\right)$$

$\rightarrow$ apply the ratio test:

$$L=\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\frac{2^{k+1}}{2^k}\cdot\cancelto{1}{\frac{(k^4+5)^{1/5}}{((k+1)^4+5)^{1/5}}}$$

$$=2$$

So that $R=\dfrac{1}{L}=\dfrac{1}{2}$

$\rightarrow$ to find the IC:

$$|x-2|<R \iff -\frac{1}{2}<x-2<\frac{1}{2}$$

$$\iff \frac{3}{2}<x<\frac{5}{2},$$

So the series definitely converges for all $x\in\left(\frac{3}{2},\frac{5}{2}\right)$. We need to check convergence at the end points when $x=\frac{3}{2}$, $x=5/2$.

$$\to f\left(\frac{3}{2}\right) = \sum_{k=1}^{\infty} \frac{2^k\left(-\frac{1}{2}\right)^k}{(k^4+5)^{1/5}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k^4+5)^{1/5}}$$

apply the AST:

- check for absolute convergence:

  does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k^4+5)^{1/5}}$ converge or not? ← a_k

  <u>intuition</u>: looks "almost" like a p-series with $p = 4/5 \leq 1$, so we expect the series to diverge.

  <u>apply the LCT</u>: compare to $b_k = \dfrac{1}{k^{4/5}}$

  $$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{4/5}}{(k^4+5)^{1/5}}$$

  $$= \lim_{k\to\infty} \frac{\cancel{k^{4/5}}^{\,1}}{\cancel{k^{4/5}}\left(1+\cancel{\frac{5}{k^4}}^{\,0}\right)^{1/5}} = 1$$

  So both series diverge.

- Check for conditional convergence:

  $\to \displaystyle\lim_{k\to\infty} \frac{1}{(k^4+5)^{1/5}} = 0$ ✓

  $\to$ check that a_k is decreasing

  $$a_{k+1} = \frac{1}{((k+1)^4+5)^{1/5}} \leq a_k = \frac{1}{(k^4+5)^{1/5}}$$ ✓

So we get conditional convergence.